

DIFFERENTIAL EQUATIONS

EXERCISE 1.5

Problems solved by;

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$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$ $\Rightarrow du = 2x dx + 8y dy = 0$ <p>for $du = 0$</p> $\Rightarrow 2x dx + 8y dy = 0$ <p style="text-align: center;"><u>Ans</u></p>	<p>Ans are exact and solve them</p> <p>① $2xy dx + x^2 dy = 0$</p> <p><u>Solution</u></p> <p>Let $M = 2xy$ & $N = x^2$.</p> <p>check for exactness.</p> $\Rightarrow \frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x} = 2x$ <p>Here this is an exact differential Eq.</p> <p>Now for solution.</p> $u(x, y) = \int M dx + k(y).$ $\Rightarrow u = \int 2xy dx + k(y).$ $\Rightarrow u = x^2 y + k(y) \quad \text{--- (1)}$ <p>differentiating w.r.t y.</p> $\Rightarrow \frac{\partial u}{\partial y} = x^2 + \frac{dk}{dy}$ <p>but $\frac{\partial u}{\partial y} = N = x^2$.</p> <p>$\therefore \frac{dk}{dy} = 0 \Rightarrow k = \text{constant} = c$</p> <p>Now the general solⁿ putting in (1)</p> $\Rightarrow u = x^2 y + c$ $\Rightarrow x^2 y = c$ <p style="text-align: center;"><u>Ans</u></p>
<p>② $u = x^2 - y^2$</p> $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$ $\Rightarrow du = 2x dx + (-2y) dy = 0$ $\Rightarrow 2x dx - 2y dy = 0$ <p style="text-align: center;"><u>Ans</u></p>	<p>② $-yx^2 dx + x^2 dy = 0$</p> <p><u>Solution</u></p> <p>Let $M = -yx^2$ and $N = x^2$.</p> $\frac{\partial M}{\partial y} = -x^2 = \frac{\partial N}{\partial x} = -x^2$ <p>Here the equation is exact.</p> <p>Now for solution.</p> $u = \int M dx + k(y)$ $\Rightarrow u = \int -yx^2 dx + k(y)$ $\Rightarrow u = -\frac{yx^3}{3} + k(y)$ <p>Now differentiating with respect to y.</p> $\Rightarrow \frac{\partial u}{\partial y} = -\frac{x^3}{1} + \frac{dk}{dy}$ <p>but $\frac{\partial u}{\partial y} = N = x^2$</p> $\therefore -\frac{x^3}{1} + \frac{dk}{dy} = x^2$ $\therefore \frac{dk}{dy} = x^2 + x^3 = \text{const.}$
<p>③ $u = e^{x^2/y}$</p> $\Rightarrow du = e^{x^2/y} \left(\frac{2x}{y} dx + (-\frac{x^2}{y^2}) dy \right) = 0$ $\Rightarrow \frac{2x}{y} e^{x^2/y} dx - \frac{x^2}{y^2} e^{x^2/y} dy = 0$ <p style="text-align: center;"><u>Ans</u></p>	<p>④ $u = \frac{1}{(x^2 + y^2)^2}$</p> $\Rightarrow du = -1(x^2 + y^2)^{-2} \cdot 2x dx + (-1(x^2 + y^2)^{-2}) \cdot 2y dy = 0$ $\Rightarrow du = \frac{-2x}{(x^2 + y^2)^2} dx - \frac{2y}{(x^2 + y^2)^2} dy = 0$ $\Rightarrow \frac{-2x}{(x^2 + y^2)^2} dx - \frac{2y}{(x^2 + y^2)^2} dy = 0$ <p style="text-align: center;"><u>Ans</u></p>
<p>⑤ $u = \tan(y^2 - x^2)$</p> $\Rightarrow du = \sec^2(y^2 - x^2) (-2x) dx + 2y \sec^2(y^2 - x^2) dy = 0$ $\Rightarrow -2x \sec^2(y^2 - x^2) dx + 2y \sec^2(y^2 - x^2) dy = 0$ <p style="text-align: center;"><u>Ans</u></p>	<p>⑥ $u = \sin x \cos y$</p> $\Rightarrow du = \cos x \cos y dx + \sin x \sin y dy = 0$ $\Rightarrow \cos x \cos y dx + \sin x \sin y dy = 0$

$$\Rightarrow \frac{\partial M}{\partial \theta} = 3e^{3\theta} = \frac{\partial N}{\partial r} = 3e^{3\theta}$$
 Hence the eq. is exact.
 Now for the sol

$$u = \int M dr$$

$$\Rightarrow u = \int e^{3\theta} dr$$

$$\Rightarrow u = e^{3\theta} \cdot r + k(\theta) \quad \text{--- (1)}$$

$$\Rightarrow \frac{\partial u}{\partial \theta} = 3e^{3\theta} \cdot r + \frac{dk}{d\theta}$$

$$\Rightarrow \frac{\partial u}{\partial \theta} = N = e^{3\theta} \cdot r = 3e^{3\theta} \cdot r + \frac{dk}{d\theta}$$

$$\Rightarrow k = \text{const.}$$
 Hence the general solution is

$$u(x, y) = re^{3\theta} = C = \text{const.}$$

Ans

$$\textcircled{11} e^{-2\theta} (rd\theta - r^2 d\theta) = 0$$

$$\Rightarrow re^{-2\theta} d\theta - r^2 e^{-2\theta} d\theta = 0$$
 Let $M = re^{-2\theta}$ & $N = -r^2 e^{-2\theta}$

$$\Rightarrow \frac{\partial M}{\partial \theta} = -2re^{-2\theta} = \frac{\partial N}{\partial r} = -2re^{-2\theta}$$
 So the equation is exact.
 Now we have

$$u = \int N d\theta + k(r)$$

$$\Rightarrow u = \int -r^2 e^{-2\theta} d\theta + k(r)$$

$$\Rightarrow u = \frac{-r^2 e^{-2\theta}}{-2} + k(r)$$

$$\Rightarrow u = \frac{r^2 e^{-2\theta}}{2} + k(r) \quad \text{--- (1)}$$

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{2re^{-2\theta}}{2} + k'(r)$$

$$\Rightarrow \frac{\partial u}{\partial r} = re^{-2\theta} + k'(r) \quad \text{--- (2)}$$
 but $\frac{\partial u}{\partial r} = M = re^{-2\theta}$ pulling in (2)

$$re^{-2\theta} = re^{-2\theta} + \frac{dk}{dr}(r)$$

$$\Rightarrow k = \text{const.}$$

$$u(x, y) = \frac{r^2 e^{-2\theta}}{2} = C = \text{const.}$$

$$\Rightarrow r^2 e^{-2\theta} = 2C$$

$$\Rightarrow r^2 e^{-2\theta} = C \quad \text{Ans}$$

$$\textcircled{12} (\cot y + x^2) dx = x \csc^2 y dy$$

$$\Rightarrow (\cot y + x^2) dx - x \csc^2 y dy = 0$$
 Let $M = \cot y + x^2$ & $N = -x \csc^2 y$

$$\Rightarrow \frac{\partial M}{\partial y} = -\csc^2 y = \frac{\partial N}{\partial x} = -\csc^2 y$$
 Hence the equation is exact.
 Now we have

$$u = \int M dx$$

$$\Rightarrow u = \int (\cot y + x^2) dx$$

$$\Rightarrow u = x \cot y + \frac{x^3}{3} + k(y) \quad \text{--- (1)}$$

$$\Rightarrow \frac{\partial u}{\partial y} = -x \csc^2 y + \frac{dk}{dy} \quad \text{--- (2)}$$
 but $\frac{\partial u}{\partial y} = N = -x \csc^2 y$
 pulling in (2)

$$\Rightarrow -x \csc^2 y = -x \csc^2 y + \frac{dk}{dy}$$

$$\Rightarrow \frac{dk}{dy} = 0$$

$$\Rightarrow k = \text{const.}$$
 Inserting into (1)
 we get the general solution as

$$u(x, y) = x \cot y + \frac{x^3}{3} = C = \text{const.}$$

Ans

Check differentiating the Ans w.r.t x

$$\Rightarrow x(-\csc^2 y) \frac{dy}{dx} + \cot y + x^2 = 0$$

$$\Rightarrow -x \csc^2 y \frac{dy}{dx} + (\cot y + x^2) = 0$$

$$\Rightarrow (\cot y + x^2) dx = x \csc^2 y dy$$
 Hence correct

Let $M = 3y^2$ & $N = x$

$\Rightarrow \frac{\partial M}{\partial y} = 6y$ & $\frac{\partial N}{\partial x} = 1$

Hence the equation is not exact. To make it exact we have to find the I.F.

And we have the formula.

$I.F = \exp \int \left(\frac{M_y - N_x}{N} \right) dx$

$\Rightarrow I.F = \exp \int \left(\frac{6y - 1}{x} \right) dx$

$\Rightarrow I.F = \exp((6y-1) \left(\frac{1}{x} \right) dx)$

$= \exp((6y-1) \ln x)$

$\Rightarrow I.F = \exp(\ln x^{6y-1})$

$\Rightarrow I.F = x^{6y-1}$

$y(1/4) = 3.8$

Let $M = 2y^2 \cos x$

& $N = -y^2 \sin x$

So that $\frac{\partial M}{\partial y} = -2y^2 \cos x$

& $\frac{\partial N}{\partial x} = -2y^2 \cos x$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ hence the equation is exact.

Now we have

$u = \int M dx$

$\Rightarrow u = \int 2y^2 \cos x dx$

$\Rightarrow u = 2y^2 \sin x + k(y)$

$\Rightarrow u = y^2 \sin x + k(y)$ (a)

differentiating w.r.t y

$\Rightarrow \frac{\partial u}{\partial y} = -y^2 \sin x + \frac{dk}{dy}$

but $\frac{\partial u}{\partial y} = N = -y^2 \sin x$

Hence $-y^2 \sin x = -y^2 \sin x + \frac{dk}{dy}$

$\Rightarrow \frac{dk}{dy} = 0$

$\Rightarrow k = \text{const.}$ Inserting into (a)

we get the solution as

$u(x, y) = y^2 \sin x = C$ (b)

now given that $y(1/4) = 3.8$

(b) $\Rightarrow (3.8)^2 \sin(1/4) = C$

$\Rightarrow C = \frac{\sin(1/4)}{3.8}$

$\Rightarrow C = \frac{1}{3.8}$ putting in (b)

$\Rightarrow y^2 \sin x = \frac{1}{3.8}$

or $y = \frac{1}{\sqrt{3.8 \sin x}}$

Let $M = x^2 + y^2$ & $N = -2xy$.
 So that $\frac{\partial M}{\partial y} = 2y \neq \frac{\partial N}{\partial x} = -2y$.
 Hence the equation is not exact.
 In order to make it exact, we have to find the I.F. i.e.

$$I.F. = \exp\left(\int \frac{M_y - N_x}{N} dx\right).$$

$$\Rightarrow I.F. = \exp\left(\int \frac{(2y + 2y)}{-2xy} dx\right).$$

$$\Rightarrow I.F. = \exp\left[\int \frac{4y}{-2xy} dx\right].$$

$$\Rightarrow I.F. = \exp\left[-\int \frac{2}{x} dx\right].$$

$$\Rightarrow I.F. = \exp[-2 \ln x].$$

$$\Rightarrow I.F. = \exp \ln x^{-2}.$$

$$\Rightarrow I.F. = x^{-2}.$$

Multiplying with (6)

$$\Rightarrow x^{-2}(x^2 + y^2) dx - 2xy \cdot x^{-2} dy = 0$$

$$\Rightarrow (1 + \frac{y^2}{x^2}) dx - \frac{2y}{x} dy = 0$$

Hence $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{2y}{x^2}$
 Hence the eq. becomes exact.
 Now we have-

$$u = \int M dx + k(y)$$

$$\Rightarrow u = \int (1 + \frac{y^2}{x^2}) dx + k(y)$$

$$\Rightarrow u = x + \left(-\frac{y^2}{x}\right) + k(y)$$

$$\Rightarrow u = x - \frac{y^2}{x} + k(y).$$

$$\Rightarrow u = \frac{x^2 - y^2}{x} + k(y). \quad \text{--- (7)}$$

$$\frac{\partial u}{\partial y} = -\frac{2y}{x} + \frac{dk}{dy}$$

but $\frac{\partial u}{\partial y} = -\frac{2y}{x} = N$.

$\Rightarrow k = 0$ inserting in (7)
 we have the solution.
 $\Rightarrow u = \frac{x^2 - y^2}{x} = C$ Ans

$\Rightarrow R = \text{const.}$ inserting in (6)
 to get the sol.

$$u(x, y) = \frac{x^2 - y^2}{x} + C$$

$$\text{or } \frac{x^2 - y^2}{x} = C. \quad \text{--- (8)}$$

Now given that $y(1) = 2$

$$\Rightarrow \frac{1 - 4}{1} = C$$

$$\Rightarrow C = -3.$$

Putting in (8)

$$\Rightarrow \frac{x^2 - y^2}{x} = -3x$$

$$\Rightarrow x^2 - y^2 + 3x = 0$$

(16) $ye^x dx + (2y + e^x) dy = 0$
 $y(0) = -1$

Let $M = ye^x$ & $N = 2y + e^x$
 So that $\frac{\partial M}{\partial y} = e^x$ and $\frac{\partial N}{\partial x} = e^x$
 Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ hence the equation is exact.
 Now for solution.

$$u = \int M dx$$

$$\Rightarrow u = \int ye^x dx.$$

$$\Rightarrow u = ye^x + k(y) \quad \text{--- (9)}$$

$$\Rightarrow \frac{\partial u}{\partial y} = e^x + \frac{dk}{dy}$$

but $\frac{\partial u}{\partial y} = N = 2y + e^x$.

Hence $e^x + \frac{dk}{dy} = 2y + e^x$

$$\Rightarrow \int \frac{dk}{dy} dy = \int 2y dy$$

$$\Rightarrow k = y^2 + C$$

inserting in (9)
 we have the solution.
 $\Rightarrow u = ye^x + y^2 = C$ Ans

Q17

$$\frac{\partial M}{\partial y} = -e^y \stackrel{?}{=} \frac{\partial N}{\partial x} = -e^y.$$

Hence the equation is exact.

Now

$$u = \int (x+1)e^x - e^y dx.$$
~~$$u = \int (x+1)e^x - e^y dx.$$~~

$$\Rightarrow u = \int (xe^x + e^x - e^y) dx.$$

$$\Rightarrow u = x^2 - \{e^x dx + e^x - xe^y + k(y)\}$$

$$\Rightarrow u = xe^x - e^x + e^y - xe^y + k(y)$$

$$\Rightarrow u = x(e^x - e^y) + k(y) \quad \text{--- (a)}$$

$$\Rightarrow \frac{\partial u}{\partial y} = -xe^y + \frac{dk}{dy}.$$

but $\frac{\partial u}{\partial y} = N = -xe^y$

$$\Rightarrow \frac{dk}{dy} = +xe^y - (-xe^y)$$

$$\Rightarrow \frac{dk}{dy} = 0 \Rightarrow k = \text{const.}$$

Putting in (a) to get the general solution as.

$$u(x,y) = x(e^x - e^y) = \text{const} = c$$

$$\Rightarrow x(e^x - e^y) = c \quad \text{--- (b)}$$

Now $y(1) = 0$.

$$(b) \Rightarrow 1(e^1 - e^0) = c$$

$$\Rightarrow e - 1 = c \quad \text{--- putting in (b)}$$

$$\Rightarrow x(e^x - e^y) = e - 1 \quad \text{Ans}$$

Q18

$$(2xy dx + dy)e^{x^2} = 0, \text{ for } x > 0$$

$$\Rightarrow \frac{e^{x^2} 2xy dx}{M} + \frac{e^{x^2} dy}{N} = 0.$$

Check

$$\frac{\partial M}{\partial y} = 2xe^{x^2} = \frac{\partial N}{\partial x} = 2xe^{x^2}.$$

Hence the eq. is exact.

$$u = \int N dy +$$

$$\Rightarrow u = \int e^{x^2} dy$$

$$\Rightarrow u = \frac{e^{x^2} y}{1} + k(x) \quad \text{--- (a)}$$

$$\Rightarrow \frac{\partial u}{\partial x} = 2xye^{x^2} + \frac{dk}{dx}$$

but $\frac{\partial u}{\partial x} = M = 2xye^{x^2}$

Hence

$$2xye^{x^2} = 2xye^{x^2} + \frac{dk}{dx}$$

$$\Rightarrow k = \text{const} \quad \text{--- putting in (a)}$$

to get

$$u(x,y) = e^{x^2} y + c.$$

or $e^{x^2} y + c = 0 \quad \text{--- (b)}$

Now $y(0) = 0$.

$$\Rightarrow e^0(0) + c = 0$$

$$\Rightarrow 0 = -c \Rightarrow c = 0 \quad \text{--- putting in (b)}$$

$$\Rightarrow e^{x^2} y = 0 \quad \text{Ans}$$

exact? (then a, b, k, l are constant)
 here the exact eq.:

$$\text{Sol} \Rightarrow \underbrace{(ax+by)}_{M_1} dx + \underbrace{(kx+ly)}_{N_1} dy = 0 \quad \text{--- (1)}$$

$$\frac{\partial M_1}{\partial y} = b \quad \& \quad \frac{\partial N_1}{\partial x} = k$$

Hence the equation could only be exact iff $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \Rightarrow b = k$.

multiplying (1)

to make it exact.

$$(ax+by)dx + (bx+ly)dy = 0$$

Now

$$u = \int (ax+by) dx$$

$$\Rightarrow u = \frac{ax^2}{2} + bxy + k(y) \quad \text{--- (2)}$$

$$\Rightarrow \frac{\partial u}{\partial y} = bx + \frac{dk}{dy}$$

$$\text{let } \frac{\partial u}{\partial y} = N = bx + ly$$

$$\therefore bx + ly = bx + \frac{dk}{dy}$$

$$\Rightarrow \frac{dk}{dy} = ly$$

$$\Rightarrow k = \frac{ly^2}{2} + c \quad \text{multiplying (2)}$$

to get the solution as

$$u(x, y) = \frac{ax^2}{2} + bxy + \frac{ly^2}{2} + c$$

$$\text{or } \frac{ax^2}{2} + bxy + \frac{ly^2}{2} = c$$

Ans

are I-F of $ydx + xdy$.

$$\frac{y}{M_1} dx + \frac{x}{N_1} dy = 0 \quad \text{--- (3)}$$

$$\frac{\partial M_1}{\partial y} = 1 \quad \& \quad \frac{\partial N_1}{\partial x} = 1$$

Hence the equation is not exact
 so lets check that whether
 $y \cdot xy^3$ & $x \cdot y^5$ are I-F or Not.

"y"

$$y^2 dx + 2xy dy = 0$$

$$\frac{\partial M_1}{\partial y} = 2y \quad \& \quad \frac{\partial N_1}{\partial x} = 2y$$

Hence the equation becomes exact.

"xy³"

$$xy^4 dx + 2x^2 y^3 dy = 0$$

$$\frac{\partial M_1}{\partial y} = 4xy^3 = \frac{\partial N_1}{\partial x} = 4xy^3$$

so exact

"x²y⁵"

$$x^2 y^5 (ydx) + x^2 y^5 (2x dy) = 0$$

$$\Rightarrow x^2 y^6 dx + 2x^3 y^5 dy = 0$$

$$\Rightarrow \frac{\partial M_1}{\partial y} = 6x^2 y^5 = \frac{\partial N_1}{\partial x} = 6x^2 y^5$$

Exact

Ans

multiplying with e^x .

$$\Rightarrow e^x \sin y dx + e^x \cos y dy = 0. \quad \text{--- (1)}$$

$$\Rightarrow \text{let } M = e^x \sin y \text{ \& } N = e^x \cos y.$$

Q25 so that- $\frac{\partial M}{\partial y} = e^x \cos y$ and $\frac{\partial N}{\partial x} = e^x \cos y$.

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \therefore \text{hence } e^x \text{ is an I.F}$$

$$\text{OR } \text{I.F} = e^{\int \frac{1}{N}(My - Nx) dx}$$

$$\Rightarrow \text{I.F} = e^{\int \frac{1}{\cos y} (\cos y - 0) dx}$$

$$\Rightarrow \text{I.F} = e^x$$

Now For solution of (1)

$$\Rightarrow u = \int M dx \Rightarrow u = \int e^x \sin y dx \Rightarrow u = e^x \sin y + k(y) \quad \text{--- (2)}$$

$$\Rightarrow \frac{\partial u}{\partial y} = e^x \cos y + \frac{dk}{dy} \text{ but } \frac{\partial u}{\partial y} = N = e^x \cos y.$$

$$\therefore e^x \cos y = e^x \cos y + \frac{dk}{dy} \Rightarrow k = \text{const.}$$

Putting in (2) to get the solution.

$$u(y) = e^x \sin y = C = \text{const.}$$

(26) $y dx + [y + \tan(x+y)] dy = 0, \cos(x+y)$

multiplying with the term / function $\cos(x+y)$.

$$\Rightarrow y \cos(x+y) dx + (y \cos(x+y) + \tan(x+y) \cos(x+y)) dy = 0$$

let $M = y \cos(x+y)$ \& $N = y \cos(x+y) + \tan(x+y) \cos(x+y)$

so that-

$$\frac{\partial M}{\partial y} = -y \sin(x+y) + \cos(x+y)$$

$$= \cos(x+y) - y \sin(x+y)$$

$$\& \frac{\partial N}{\partial x} = -y \sin(x+y) + \sec^2(x+y) \cos(x+y) + \tan(x+y) (-\sin(x+y))$$

$$= -y \sin(x+y) + \sec^2(x+y) - \sin^2(x+y) \sec^2(x+y)$$

$$= -y \sin(x+y) + \sec^2(x+y) (1 - \sin^2(x+y))$$

$$= -y \sin(x+y) + \sec^2(x+y) \cos^2(x+y)$$

$$\Rightarrow \frac{\partial N}{\partial x} = -y \sin(x+y) + \cos(x+y)$$

since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ hence the equation is

$$u = \int M dx \Rightarrow u = \int y \cos(x+y) dx$$

$$\Rightarrow u = +y \sin(x+y) + k(y) \quad \text{--- (1)}$$

Differentiating w.r.t y we get.

$$\frac{\partial u}{\partial y} = \sin(x+y) + y \cos(x+y) + \frac{dk}{dy}$$

$$\text{but } \frac{\partial u}{\partial y} = N = y \cos(x+y) + \cos(x+y) \frac{\sin(x+y)}{\cos(x+y)} \\ = y \cos(x+y) + \sin(x+y)$$

$$\therefore y \cos(x+y) + \sin(x+y) = \sin(x+y) + y \cos(x+y) + \frac{dk}{dy}$$

$\Rightarrow k = \text{const}$ putting in (1) to get the solution as

$$u(x+y) = y \sin(x+y) = \text{const} = c$$

check $du = M dx + N dy$

$$\Rightarrow du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = y \cos(x+y) dx + (y \cos(x+y) + \sin(x+y)) dy$$

verified

$$(2) (a+1)y dx + (b+1)x dy = 0 \quad \text{--- (1)} \quad x^a y^b$$

Here I.F. = $x^a y^b$ multiplying with the eq (1)

$$\Rightarrow x^a y^b (a+1)y dx + x^a y^b (b+1)x dy = 0$$

$$\Rightarrow x^a y^{b+1} (a+1) dx + x^{a+1} y^b (b+1) dy = 0$$

$$\text{Now let } M = x^a y^{b+1} (a+1) \text{ \& } N = x^{a+1} y^b (b+1)$$

$$\text{to check } \frac{\partial M}{\partial y} = (a+1)(b+1)x^a y^b = \frac{\partial N}{\partial x} = (a+1)(b+1)x^a y^b$$

Now in order to get the solution solve.

$$u = \int M dx \Rightarrow u = \int (a+1)x^a y^{b+1} dx$$

$$\Rightarrow u = (a+1)(y^{b+1}) \frac{(x^{a+1})}{a+1} + k(y)$$

$$\Rightarrow u = (y^{b+1})(x^{a+1}) + k(y) \quad \text{--- (2)}$$

$$\Rightarrow \frac{\partial u}{\partial y} = (b+1)(y^b)(x^{a+1}) + \frac{dk}{dy}$$

$$\text{but } \frac{\partial u}{\partial y} = N = (b+1)x^{a+1}y^b$$

$$\therefore (a+1)(y^{b+1})(x^{a+1}) = (b+1)y^b(x^{a+1}) + \frac{dk}{dy}$$

check

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$\Rightarrow du = (a+1)x^a y^{b+1} dx + (b+1)y^b x^{a+1} dy$$

Q38 $3(y+1)dx = 2x dy$ — (1), $(y+1)/x^4$

Ans I.F = $(y+1)/x^4$ multiplying with — (1)

$$\Rightarrow \frac{3(y+1)^2}{x^4} dx - \frac{2(y+1)}{x^3} dy = 0$$

Let $M = \frac{3(y+1)^2}{x^4}$ & $N = -\frac{2(y+1)}{x^3}$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{6(y+1)}{x^4} = \frac{\partial N}{\partial x} = \frac{6(y+1)}{x^4}$$

Hence the equation is exact now

In order to find its solution, we have

$$u = \int M dx \Rightarrow u = \int \frac{3(y+1)^2}{x^4} dx$$

$$\Rightarrow u = -\frac{(y+1)^2}{x^3} + k(y) \quad \text{--- (2)}$$

Now $\frac{\partial u}{\partial y} = -\frac{2(y+1)}{x^3} + \frac{dk}{dy}$

but $\frac{\partial u}{\partial y} = N = -\frac{2(y+1)}{x^3} = -\frac{2y+2}{x^3} + \frac{dk}{dy}$

$$\Rightarrow \frac{dk}{dy} = 0 \Rightarrow k = \text{const.}$$

Putting in (2) to get the

solution as

$$u(x,y) = -\frac{(y+1)^2}{x^3} = c = \text{const.}$$

check

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$$

$$\Rightarrow du = \frac{3(y+1)^2}{x^4} dx - \frac{2(y+1)}{x^3} dy = 0$$

we get

$$\text{Q29 } \left(\frac{x}{y} + 1\right) dx + \frac{2}{y} dy = 0$$

let $M = \frac{x}{y} + 1$, $N = \frac{2}{y}$

$$\Rightarrow \frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x} = 0 \quad \text{hence it is the integrating factor.}$$

now $u = \int M dx \Rightarrow u = \int \left(\frac{x}{y} + 1\right) dx$

$$\Rightarrow u = \frac{1}{2} \ln x + x + \text{fnc}(y) \quad \text{--- (2)}$$

$$\Rightarrow \frac{\partial u}{\partial y} = \text{fnc}'(y) \quad \text{but } \frac{\partial u}{\partial y} = \frac{2}{y} = \frac{d(\text{fnc})}{dy}$$

$$\Rightarrow \frac{d(\text{fnc})}{dy} = \frac{2}{y} \Rightarrow \int \frac{d(\text{fnc})}{dy} dy = \int \frac{2}{y} dy$$

$$\Rightarrow \text{fnc} = 2 \ln y + \text{cnc} \quad \text{putting in (2) to get the solution as}$$

$$u(x,y) = \frac{1}{2} \ln x + x + 2 \ln y = \text{cnc}$$

$$\text{or } \ln x^{\frac{1}{2}} + \ln y^2 + x = \text{cnc}$$

$$\Rightarrow \frac{\ln x^{\frac{1}{2}}}{e} = -x \Rightarrow \frac{x^{\frac{1}{2}}}{e} = e^{-x}$$

$$\Rightarrow x^{\frac{1}{2}} e^x = C \quad \text{Ans}$$

Q30 $2 \cos y dx = \tan 2x \sin y dy$, $\cos 2x$

$$\Rightarrow 2 \cos y dx - \tan 2x \cdot \sin y dy = 0$$

Multiplying with $\cos 2x$

$$\Rightarrow 2 \cos 2x \cos y dx - \cos 2x \tan 2x \cdot \sin y dy = 0$$

$$\Rightarrow 2 \cos 2x \cos y dx - \sin 2x \sin y dy = 0$$

now let $M = 2 \cos 2x \cos y$ & $N = -\sin 2x \sin y$

$$\Rightarrow \frac{\partial M}{\partial y} = -2 \cos 2x \sin y = \frac{\partial N}{\partial x} = -2 \sin 2x \sin y$$

Since the equation becomes exact, hence it is the Integrating Factor.

now solve the Remaining Part of the the Question to get the solution yourself.

let $M = 2\cosh x \cos y$ & $N = -\sinh x \sin y$, so that
 $\frac{\partial M}{\partial y} = -2\cosh x \sin y$ & $\frac{\partial N}{\partial x} = -\cosh x \sin y$.

Q3) Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, hence the equation is not exact.
 In order to make it exact we have to find its integrating factor.

$$\begin{aligned} I.F. &= \exp \int \frac{1}{N} (M_y - N_x) dx \\ \Rightarrow I.F. &= \exp \int \left(\frac{-2\cosh x \sin y + \cosh x \sin y}{-\sinh x \sin y} \right) dx \\ \Rightarrow I.F. &= \exp \int \frac{-\cosh x \sin y}{-\sinh x \sin y} dx \\ \Rightarrow I.F. &= \exp \int \cot x dx. \end{aligned}$$

$$\Rightarrow I.F. = \exp \cdot \ln(\sinh x) = e^{\ln(\sinh x)} = \sinh x.$$

now multiplying this with (1).

$$\begin{aligned} u &= \int M dx \\ \Rightarrow \int \sinh x \cdot 2\cosh x \cos y dx &= \int \sinh x \cdot \cosh x \cdot 2 \cos y dx \\ &= \frac{1}{2} \int \sinh(2x) \cdot 2 \cos y dx \\ &= \frac{1}{2} \int \sinh(2x) dx \cdot 2 \cos y \\ &= \frac{1}{2} \cdot \frac{1}{2} \sinh(2x) \cdot 2 \cos y \\ &= \frac{1}{2} \sinh(2x) \cos y \end{aligned}$$

$$\textcircled{1} \Rightarrow 2\sinh x \cosh x \cos y dx - \sinh x \sin y dy$$

so that $\frac{\partial M}{\partial y} = -2\sinh x \sin y$ & $\frac{\partial N}{\partial x} = -\sinh x \cos y$

$$\text{as } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ so the equation is now exact.}$$

Now

$$\begin{aligned} u &= \int M dx \Rightarrow u = \int 2\sinh x \cosh x \cos y dx \\ \Rightarrow u &= \int 2\sinh x \cosh x dx \cdot \cos y \\ &= 2 \cos y \int f(x) \cdot f'(x) dx \\ \Rightarrow u &= \frac{1}{2} \sinh^2 x \cdot 2 \cos y = \cos y \sinh^2 x + k(y) \end{aligned}$$

$\Rightarrow k = \text{const}$ pulling m (2) to get the general solution as

$$u(x, y) = \cosh(\sinh^2 x) = c = \text{const}$$

Check

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$\Rightarrow du = 2 \cosh(\sinh^2 x) \cosh x dx + \sinh^2 x (-\sinh y) dy$$

Ans

32) $2xy dx + 3x^2 dy = 0$ — (1)

For I.F we have the formula

$$I.F = \exp \left\{ \int \frac{1}{M} (My - Nx) dx \right\}$$

but we can also find the I.F by inspection.

multiply (1) with $\frac{1}{x^2 y}$

$$\Rightarrow \frac{2xy}{x^2 y} dx + \frac{3x^2}{x^2 y} dy = 0$$

$$\Rightarrow \frac{2}{x} dx + \frac{3}{y} dy = 0 \quad \text{since } \frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x} = 0$$

now for the sol.

$$u = \int \frac{2}{x} dx \Rightarrow u = 2 \ln x + k(y)$$

$$\Rightarrow u = 2 \ln x + ky \quad \text{--- (2)}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{dk}{dy} \Rightarrow \frac{\partial u}{\partial y} = N = \frac{3}{y} = \frac{dk}{dy}$$

$$\Rightarrow \int \frac{dk}{dy} dy = \int \frac{3}{y} dy$$

$\Rightarrow k = 3 \ln y + c \Rightarrow$ putting m (2) to get the solution as

$$u(x, y) = 2 \ln x + 3 \ln y = c$$

$$\text{or } \ln x^2 + \ln y^3 = c \quad \text{--- (3)}$$

$$\Rightarrow x^2 y^3 = e^c \Rightarrow x^2 y^3 = c \quad \text{--- (4) Ans}$$

$$(33) (2xy + 4x^2)dx = x \sin y dy \quad \text{--- (1)}$$

$$\Rightarrow (2xy + 4x^2)dx - x \sin y dy = 0 \quad \text{--- (1)}$$

$$\Rightarrow I.F = \exp \int \frac{-2 \sin y + \sin y}{-x \sin y} dx = \exp \int \frac{+\sin y}{+x \sin y} dx$$

$$\Rightarrow I.F = \exp(+\ln x) = x^{+1} = x = \frac{1}{x^{-1}} \quad \text{multiplying with (1)}$$

$$\Rightarrow \frac{2xy}{x^{-1}} + \frac{4x^2}{x^{-1}} dx - \frac{x \sin y}{x^{-1}} dy = 0 \Rightarrow 2x^2 \cos y + 4x^3 dx - x \sin y dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} (2x^2 \cos y) = -2x^2 \sin y = \frac{\partial N}{\partial x} = \frac{\partial}{\partial x} (-x \sin y) = -x \sin y$$

solve yourself.

$$(34) 2x \cos y dx - \sin y dy = 0 \quad \text{--- (1)}$$

$$I.F = \exp \int \frac{+2 \cos y - 0}{-\sin y} dx = \exp \int 2 dx = \exp(2x) = e^{2x} \quad \text{multiplying with (1)}$$

$$\Rightarrow 2e^{2x} \cos y dx - e^{2x} \sin y dy = 0$$

$$\frac{\partial M}{\partial y} = -2e^{2x} \sin y = \frac{\partial N}{\partial x} = -2e^{2x} \sin y$$

so the equation now becomes exact.

next solve yourself.

$$(35) 2xe^x \tan y dx + \sec^2 y dy = 0 \quad \text{--- (1)}$$

$$I.F = \exp \int \frac{2x \sec^2 y - 0}{\sec^2 y} dx = \exp \int 2x dx = e^{x^2} \quad \text{multiplying with (1)}$$

$$\Rightarrow 2xe^{x^2} \tan y dx + e^{x^2} \sec^2 y dy = 0$$

$$\Rightarrow \frac{\partial M}{\partial y} = 2xe^{x^2} \sec^2 y = \frac{\partial N}{\partial x} = 2xe^{x^2} \sec^2 y$$

now the equation is exact.

for solution

$$u = \int 2xe^{x^2} \tan y dx$$

and $x^2 = t \Rightarrow dt = 2x dx \Rightarrow \frac{dt}{2} = x dx$

$$\Rightarrow I = \int \frac{e^t}{2} dt = \frac{1}{2} e^t = \frac{1}{2} e^{x^2} + k(y) \quad \text{pulling in (2)}$$

$$\Rightarrow u = 2 \tan y \cdot \left(\frac{1}{2} e^{x^2} \right) + k(y)$$

$$\Rightarrow u = e^{x^2} \tan y + k(y) \quad \text{--- (3)}$$

$$\Rightarrow \frac{\partial u}{\partial y} = e^{x^2} \sec^2 y + \frac{dk}{dy}$$

$$\text{but } \frac{\partial u}{\partial y} = e^{x^2} \sec^2 y = N$$

$$\Rightarrow \frac{\partial u}{\partial y} = e^{x^2} \sec^2 y = e^{x^2} \sec^2 y + \frac{dk}{dy} \Rightarrow k = \text{const}$$

pulling in (3)
we get

$$u(x, y) = e^{x^2} \tan y = C = \text{const}$$

Ans.

(36) $(y+1)dx - (x+1)dy = 0$ --- (1)

$$I.F = \exp \int \frac{1+y}{-(x+1)} dx = \int \frac{-2}{-(x+1)} dx = e^{-2 \int \frac{1}{x+1} dx}$$

$$\Rightarrow I.F = e^{-2 \ln(x+1)} = (x+1)^{-2} \text{ multiplying with (1)}$$

$$\Rightarrow \frac{1}{(x+1)^2}$$

$$(x+1)^{-2} (y+1)dx - (x+1)^{-2} (x+1)dy = 0$$

$$\Rightarrow \frac{(y+1)}{(x+1)^2} dx - (x+1)^{-1} dy = 0.$$

this equation is now exact.

$$\text{Let } \frac{1}{(x+1)^2} = \frac{1}{(x+1)^2}$$

For solution we have

$$u = \int \frac{y+1}{(x+1)^2} dx \Rightarrow u = -\frac{(y+1)}{(x+1)} + k(y) \quad \text{--- (2)}$$

$$\Rightarrow \frac{\partial u}{\partial y} = -\frac{1}{(x+1)} + \frac{dk}{dy} \quad \text{--- (3)}$$

$$\text{but } \frac{\partial u}{\partial y} = N = -\frac{1}{(x+1)} = -\frac{1}{(x+1)} + \frac{dk}{dy}$$

$$\Rightarrow k = \text{const} \text{ pulling in (2) to get the sol as}$$

$$u(x, y) = -\frac{(y+1)}{(x+1)} = c = \text{const.}$$

$$(37) \quad x^{-1} \cos y \, dx + \sin y \, dy = 0 \quad \text{--- (1)}$$

$$I \cdot F = \exp \int \frac{x^{-1} \sin y - 0}{\sin y} \, dx = \exp(\ln x) = x.$$

multiplying with (1)

$$\Rightarrow \underbrace{\cos y \, dx}_M + \underbrace{x \sin y \, dy}_N = 0 \quad \text{--- (2)}$$

This Eq. is now exact. i.e. $\frac{\partial M}{\partial y} = \sin y = \frac{\partial N}{\partial x}$
For solution we have the formula.

$$u = \int \cos y \, dx.$$

$$\Rightarrow u = x \cos y + k(y) \quad \text{--- (3)}$$

$$\Rightarrow \frac{\partial u}{\partial y} = x \sin y + \frac{dk}{dy}, \text{ but } \frac{\partial u}{\partial y} = N = x \sin y.$$

$$\therefore x \sin y = x \sin y + \frac{dk}{dy} \Rightarrow k = \text{const. putting in (3)}$$

\therefore to get the solution as

$$u(x, y) = x \cos y = c = \text{const.}$$

Read Prob. (39)

(*) LINEAR DIFFERENTIAL EQUATION & BERNOULLI EQUATION

A first order differential equation of the form
(1) $- y' + P(x)y = r(x)$ is called as linear.

For a differential equation to be linear it is must not it ~~should~~ ^{can} be written as in (1).

or simply, A first order differential equation is said to be linear, if it can be written as

$$y' + P(x)y = r(x) \quad \text{--- (1)}$$

This eq is linear in the unknown function y and y' , whereas P as well as r on the right may be any given functions of x .

If the right side $r(x)$ is zero for all x in the interval in which we consider the equation.

($r(x) = 0$), the the equation is said to be **homogeneous**; otherwise it is said to be **non-homogeneous**.

Let us find a formula for the general solution of (2) in some interval I , assuming that P and r are continuous in I , for the homogeneous equation.

$$y' + P(x)y = 0 \quad \Rightarrow r(x) = 0$$

$$\Rightarrow \frac{dy}{dx} + P(x)y = 0$$

$$\Rightarrow dy + yP(x)dx = 0 \Rightarrow \frac{dy}{y} + Pdx = 0$$

$$\Rightarrow \frac{dy}{y} = -Pdx \Rightarrow \ln y = -\int Pdx + c$$

$$\Rightarrow y = ce^{-\int Pdx + c} \quad \text{by taking exponentials on both sides}$$

$$\Rightarrow y = ce^{-\int Pdx} \quad \text{--- (2) (} c = \pm e^c \text{ when } y \neq 0 \text{)}$$

here we may also take $c=0$ to obtain the

The Equation .

$$\frac{dy}{dx} + yP(x) = Q(x) \quad \text{--- (1)}$$

whose left member is linear in both the dependent variables and its derivative, is called a linear equation of 1st order.

e.g

$\frac{dy}{dx} + 3xy = \sin x$ is called linear while

$\frac{dy}{dx} + 3xy^2 = \sin x$ is not.

Solution of Nonhomogeneous Linear Equation

Let us now consider the nonhomogeneous equation (1)

$$y' + yP(x) = Q(x)$$

$$\Rightarrow \frac{dy}{dx} + (yP_x - yx) = 0 \Rightarrow dy + (Py - 1)dx = 0 \quad \text{--- (2)}$$

this is $Pdx + Qdy = 0$, where $P = Py - 1$ and $Q = 1$.
Since it is not exact so its Integrating factor is

$$I.F = \exp \int \frac{Py - 1}{1} dx = \exp \int P dx.$$

Multiplying with (2) to make the eq (2) exact.

$$\Rightarrow e^{\int P dx} dy + (Py - 1)e^{\int P dx} dx = 0$$

$$\Rightarrow e^{\int P dx} \frac{dy}{dx} + (Py - 1)e^{\int P dx} = 0$$

$$\Rightarrow e^{\int P dx} \left(\frac{dy}{dx} + Py \right) = 1e^{\int P dx}$$

$$\Rightarrow e^{\int P dx} \frac{dy}{dx} + yPe^{\int P dx} = 1e^{\int P dx}$$

$$\Rightarrow \frac{d}{dx} (e^{\int P dx} \cdot y) = 1e^{\int P dx}$$

$$\Rightarrow e^{\int P dx} \cdot y = \int 1e^{\int P dx} dx + C$$

$$y = e^{-\int P dx} \left[\int e^{\int P dx} dx + C \right] \quad \text{--- (3)}$$

now let $h = \int p dx$.

so (4) can be written as,

$$y = e^{-h} \left[\int r e^h dx + c \right] \text{ --- (5)}$$

This represents the general sol of eq (1)

EXAMPLE 1

$$y' - y = e^{2x}$$

here $p = -1$, $r = e^{2x}$, $h = \int p dx = \int -1 dx = -x$

$$\therefore y = e^{-h} \left[\int e^h r dx + c \right] = e^{+x} \left[\int e^{-x} e^{2x} dx + c \right]$$

$$\Rightarrow y = e^x \left[\int e^x dx + c \right] = e^{2x} + c e^x.$$

Ans.

In simpler cases, such as the present, we may not need the general formula (5) but may wish to proceed directly, multiplying the given equation by $e^h = e^{-x}$, thus gives,

$$(y' - y)e^{-x} = (ye^{-x})' = e^{2x-x} = e^x.$$

Integrating on both sides, we obtain the same result as before.

$$ye^{-x} = e^x + c. \Rightarrow y = e^{2x} + ce^x.$$

Ans.

If the Integral cannot be integrated by the usual methods of calculus (as often in practice) we may have to use a numerical methods for integrals or for differential equations itself.

(*) BERNOULLI'S EQUATION (Reduction to Linear Form)

An equation of the form.

$$y' + (y)P(x) = y^a Q(x) \quad \text{--- (1)}$$

or $y^{-a} \frac{dy}{dx} + y^{1-a} P(x) = Q(x)$ where $a \neq 0, 1$

is called Bernoulli's Equation.

This is reduced to the form, namely.

$$u' + (1-a)Pu = (1-a)Q \quad \text{where } u = [y(x)]^{1-a}$$

In (1) if $a = 0$ or $a = 1$ then the equation becomes linear otherwise non-linear.

We now prove the formula for the non-linear equation.

In (1) Let $y^{1-a} = u(x)$

$$\Rightarrow u' = (1-a)y^{-a} \frac{dy}{dx}$$

Putting $\frac{dy}{dx} = y' = y^a Q(x) - yP(x)$ from (1) we get

$$u' = (1-a)y^{-a} (y^a Q(x) - yP(x))$$

$$\Rightarrow u' = (1-a)(Q(x) - y^{1-a}P(x))$$

$$\Rightarrow u' + (y)^{1-a}P(x) = (1-a)Q(x)$$

$$\Rightarrow u' + uP(x)(1-a) = (1-a)Q(x)$$

$$\Rightarrow u' + (1-a)Pu = (1-a)Q \quad \text{--- (2)}$$

By (5)

$$u = e^{\int p dx} \left[\int e^{-\int p dx} \cdot r dx + c \right]$$

where $P = (1-a)P$ & $r = (1-a)Q$.

$\frac{dy}{dx} = e^{-\ln(\sec x)} = e^{\ln x} \Rightarrow e^{\ln x} = x = 1/x$
 $\int e^{-\ln(\sec x)} = \cos x$
 $e^{-\ln(\sec x)} = e^{\ln(\sec x)^{-1}} = (\sec x)^{-1} = \frac{1}{\sec x} = \cos x$

②

⊛ General Solutions: In Probs 3-14, find the general solutions of the following differential Equations

③ $y' - y = 4$
 here we use the formula (4) i.e
 $y = e^{-\int p dx} \left[\int e^{\int p dx} \cdot r + c \right] \quad \text{--- (4) where } h = \int p dx$
 here $p = -1$, $r = 4$ & $h = \int p dx = -x +$
 putting values in (4)
 $\Rightarrow y = e^x \left[\int e^{-x} \cdot 4 + c \right]$
 $\Rightarrow y = -e^x e^{-x} \cdot 4 + ce^x$
 $\Rightarrow y = -4 + ce^x$ Ans

④ $y' + 2y = 2.5$
 here $p = 2$, $r = 2.5$, $h = \int 2 dx = 2x$ so
 (4) $\Rightarrow y = e^{-2x} \left[\int e^{2x} \cdot 2.5 + c \right] \Rightarrow y = e^{-2x} \left[\frac{e^{2x}}{2} (2.5) + c \right]$
 $\Rightarrow y = ce^{-2x} + 2.5/2$ Ans

⑤ $y' + 3xy = 0$
 here $p = 3x$, $r = 0$, $h = \int 3x dx = \frac{3}{2}x^2$
 $\Rightarrow y = e^{-\frac{3}{2}x^2} \left[\int 0 \cdot e^{\frac{3}{2}x^2} dx + c \right] = ce^{-\frac{3}{2}x^2}$ Ans

Here $P=4$, $r=e^{-4x}$, $h=4x$.

Q1 $\Rightarrow y = e^{-4x} \left[\int e^{4x} \cdot e^{-4x} dx + C \right]$

$\Rightarrow y = e^{-4x} [x + C]$

$\Rightarrow y = x e^{-4x} + C e^{-4x}$

Ans

⑧ $y' + 4y = \cos x$

Here $P=4$, $r = \cos x$, $h=4x$.

$\Rightarrow y = e^{-4x} \left[\int e^{4x} \cos x dx + C \right]$

$\Rightarrow y = e^{-4x} [I + C] \quad \text{--- (1)}$

Now $I = \int e^{4x} \cos x dx$

$\Rightarrow I = \frac{e^{4x}}{4} \cos x - \int \frac{e^{4x}}{4} (-\sin x) dx$

$\Rightarrow I = \frac{e^{4x}}{4} \cos x + \left(\frac{e^{4x}}{16} \sin x - \frac{1}{4} \int \frac{e^{4x}}{4} \cos x dx \right)$ here $P=1$ so $h=x$ and $r=e^{-x} \tan x$.

$\Rightarrow I = \frac{e^{4x}}{4} \cos x + \frac{e^{4x}}{16} \sin x - \frac{1}{16} \int e^{4x} \cos x dx$ so $y = e^{-x} \left[\int e^x e^{-x} \tan x dx + C \right]$

$\Rightarrow I = \frac{e^{4x}}{4} \cos x + \frac{e^{4x}}{16} \sin x - \frac{1}{16} I$

$\Rightarrow \left(1 + \frac{1}{16}\right) I = \frac{e^{4x}}{4} \cos x + \frac{e^{4x}}{16} \sin x$

$\Rightarrow \frac{17}{16} I = \frac{e^{4x}}{4} \cos x + \frac{e^{4x}}{16} \sin x$

$\Rightarrow I = \frac{4e^{4x}}{17} \cos x + \frac{e^{4x}}{17} \sin x$

Putting in (1)

$\Rightarrow y = e^{-4x} \left[\frac{4e^{4x}}{17} \cos x + \frac{e^{4x}}{17} \sin x + C \right]$

$\Rightarrow y = \frac{4}{17} \cos x + \frac{\sin x}{17} + C e^{-4x}$

Ans

⑨ $y' = 2y/x + x^2 e^x$ Eq

$\Rightarrow y' - 2y/x = x^2 e^x$

Here $P = -2/x$, so $h = \int -2/x dx$

$\Rightarrow h = -2 \ln x = \ln x^{-2}$

$r = x^2 e^x$

$\Rightarrow y = e^{-\ln x^{-2}} \left[\int e^{\ln x^{-2}} \cdot x^2 e^x dx + C \right]$

$\Rightarrow y = x^2 \left[\int x^2 e^x dx + C \right]$

$\Rightarrow y = x^2 \left[\int e^x dx + C \right]$

$\Rightarrow y = x^2 e^x + C x^2$ Ans

⑩ $y' + y = e^x \tan x$

Here $P=1$ so $h=x$ and $r=e^{-x} \tan x$.

$\Rightarrow y = e^{-x} \left[\int e^x e^{-x} \tan x dx + C \right]$

$\Rightarrow y = e^{-x} \left[\int \tan x dx + C \right]$

$\Rightarrow y = e^{-x} (\ln |\sec x| + C)$ Ans

⑪ $y' = (y-2) \cot x$

$\Rightarrow y' - y \cot x = -2 \cot x$

Here $P = -\cot x$, $h = \int -\cot x dx$

so $h = -\ln |\csc x| = \ln |\csc x|$

$\& e^{-h} = e^{-\ln |\csc x|} = \sin x$

$r = -2 \cot x$

$\Rightarrow y = \sin x \left[\int \frac{-2 \cot x}{\sin x} dx + C \right]$

$\Rightarrow y = \sin x \left[-2 \int \frac{\cos x}{\sin^2 x} dx + C \right]$

$\Rightarrow y = -2 + C \sin x$ Ans

$$\Rightarrow y' + 3y/x = 1/x^4$$

Here $P = 3/x$ so $h = 3 \ln x$.
 and $e^{-h} = x^{-3}$, $e^{-h} = x^{-3}$.

$$r = 1/x^4$$

$$\Rightarrow y = x^{-3} \left[\int x^3 \cdot \frac{1}{x^4} dx + c \right]$$

$$\Rightarrow y = x^{-3} [\ln x + c]$$

Ans

$$(13) y' + y \sin x = e^{\cos x}$$

Here $P = \sin x$ hence $h = -\cos x$.
 $e^h = e^{-\cos x}$ & $e^{-h} = e^{\cos x}$.
 $r = e^{\cos x}$.

$$\Rightarrow y = e^{\cos x} \left[\int e^{-\cos x} e^{\cos x} dx + c \right]$$

$$\Rightarrow y = e^{\cos x} [x + c]$$

Ans

$$(14) x^2 y' + 2xy = 8 \sinh 5x$$

$$\Rightarrow y' + 2y/x = \frac{1}{x^2} \sinh 5x$$

Here $P = 2/x$ so $h = 2 \ln x$
 $\Rightarrow e^h = x^2$ and $e^{-h} = x^{-2}$.
 $r = \frac{1}{x^2} \sinh 5x$.

$$\Rightarrow y = x^{-2} \left[\int \frac{x^2 \sinh 5x}{x^2} dx + c \right]$$

$$\Rightarrow y = x^{-2} \left[\int \sinh 5x dx + c \right]$$

$$\Rightarrow y = x^{-2} \left[\frac{\cosh 5x}{5} + c \right]$$

Ans

Solve the following initial value problems.

$$(15) y' + 4y = 20, y(0) = 2$$

Here $P = 4$, $h = 4x$.
 $e^h = e^{4x}$, $e^{-h} = e^{-4x}$
 $r = 20$.

$$\Rightarrow y = e^{-4x} \left[\int e^{4x} \cdot 20 dx + c \right]$$

$$\Rightarrow y = e^{-4x} \left[\frac{20e^{4x}}{4} + c \right]$$

$$\Rightarrow y = e^{-4x} [5e^{4x} + c]$$

$$\Rightarrow y = 5 + ce^{-4x} \text{ (general sol.)}$$

(1) Ans

Now $y(0) = 2$.

$$(1) \Rightarrow 2 = 5 + c$$

$$\Rightarrow c = -3 \text{ putting in (1)}$$

$$\Rightarrow y = 5 - 3e^{-4x}$$

Ans

$$(16) y' - (1+3x^{-1})y = x+2$$

x $y(0) = e-1$

Here $P = -(1+3x^{-1})$
 $\Rightarrow h = -3 \ln x - x$

$$\Rightarrow e^h = x^{-3}, e^{-h} = x^3$$

$$r = x+2$$

$$\Rightarrow y = x^{-3} \left[\int x^3 (x+2) dx + c \right]$$

$$\Rightarrow y = x^{-3} \left[\int (x^4 + 2x^3) dx + c \right]$$

$$\Rightarrow y = x^{-3} \left[\frac{x^5}{5} + \frac{x^4}{2} + c \right]$$

$$\Rightarrow y = \frac{x^2}{5} + \frac{x}{2} + cx^{-3}$$

X

Ans

Here $P = -2 \tanh(2x) \Rightarrow h = \int P dx = \int -2 \tanh(2x) dx$

Q11 $\Rightarrow h = -\ln \cosh(2x)$ so $e^h = \cosh^{-1}(2x) = \operatorname{sech}(2x)$
 $\& e^{-h} = \cosh(2x)$

$y = -2 \tanh(2x)$ sech 2x

$\therefore y = \cosh(2x) \left[\int \operatorname{sech}(2x) \cdot 2 \tanh(2x) dx + C \right]$

$\Rightarrow y = \cosh(2x) \left[\int \operatorname{sech}^2(2x) \sinh(2x) dx + C \right]$

$\Rightarrow y = \cosh(2x) \left(-2 \tanh(2x) \cdot \sinh(2x) - \int \tanh(2x) \cdot \cosh(2x) dx \right) + C$

$\Rightarrow y = \cosh(2x) \left(-2 \tanh(2x) \sinh(2x) + 2 \int \sinh(2x) dx + C \right)$

$\Rightarrow y = \cosh(2x) \left(-2 \tanh(2x) \sinh(2x) + 2 \cosh(2x) + C \right) \quad \text{--- (1)}$

Now $y(0) = 4$ *when x=0*

$\Rightarrow 4 = \cosh(0) (0 + 2 \cosh(0) + C)$

$\Rightarrow 4 = 1 + C \Rightarrow C = 3$ *putting in (1)*

$\Rightarrow y = (-\cosh 2x) (2 \tanh 2x \sinh 2x) + 2 \cosh^2 2x + 3 \cosh 2x$

$\Rightarrow y = -\sinh^2 2x + 2 \cosh^2 2x + 3 \cosh 2x$

$\Rightarrow y = 1 + 3 \cosh 2x$ Ans

(16) $y' - (1 + 3/x)y = x + 2, \quad y(1) = e - 1$

$\Rightarrow y' - (1 + 3/x)y = x + 2$

Here $P = -(1 + 3/x) \Rightarrow h = \int P dx = -\int (1 + 3/x) dx$

$\Rightarrow h = -[x + 3 \ln x] \Rightarrow h = -(x + 3 \ln x)$

$\therefore e^h = e^{-(x + 3 \ln x)} = e^{-x} \cdot e^{-3 \ln x} = x^{-3} e^{-x}$

$\& e^{-h} = e^x e^{3 \ln x} = e^x x^3 = x^3 e^x$

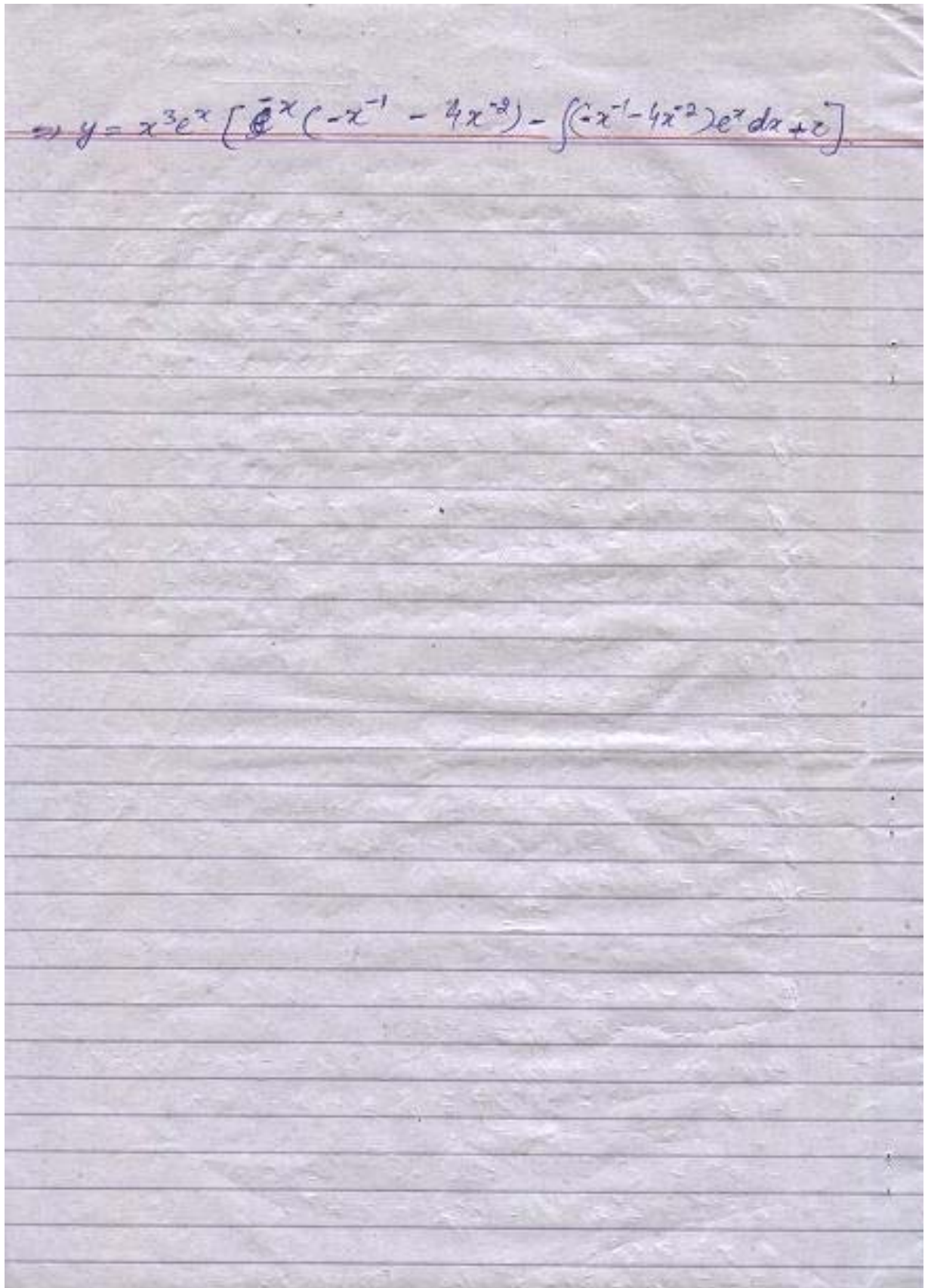
$\& r = x + 2$

According to the formula,

$\therefore y = x^3 e^x \left[\int x^{-3} e^{-x} (x + 2) dx + C \right]$

$\Rightarrow y = x^3 e^x \left[(x^{-2} e^{-x} + 2x^{-3} e^{-x}) dx + C \right]$

$\Rightarrow y = x^3 e^x \left[\int x^{-2} (x^{-2} + 2x^{-3}) dx + C \right]$



A photograph of a piece of lined paper with a handwritten mathematical equation. The equation is written in blue ink and is the only content on the page. It represents a solution for y in terms of x, involving an exponential function and an integral.

$$\Rightarrow y = x^3 e^x \left[\cancel{e^x} (-x^{-1} - 4x^{-2}) - \int (-x^{-1} - 4x^{-2}) e^x dx + C \right]$$

$$y' + 2y = e^x (3 \sin 2x + 2 \cos 2x)$$

$$P = 2, \quad Q(x) = e^{2x} (3 \sin 2x + 2 \cos 2x)$$

$$u = \int P dx = 2x$$

$$y = e^{-2x} \left[\int e^{2x} \cdot e^x (3 \sin 2x + 2 \cos 2x) dx + C \right]$$

$$A = 3e^{3x} \sin 2x$$

$$= e^{3x} \sin 2x - \int e^{3x} \cdot 2 \cos 2x dx$$

$$= e^{3x} \sin 2x - \frac{2e^{3x}}{3} \cos 2x - \frac{4}{3} \int e^{2x} \sin 2x$$

$$A + \frac{4}{3} A = \frac{e^{3x}}{3} [3 \sin 2x + 2 \cos 2x]$$

$$\Rightarrow A = \frac{e^{3x}}{13} [3 \sin 2x + 2 \cos 2x]$$

$$\text{Q21} \quad y' = 1 + y^2$$

$$y' =$$

$$\frac{dy}{1+y^2} = dx$$

$$\tan^{-1} y = x + C$$

$$y+1 = y^2$$

$$u = y^2 = y'$$