

DIFFERENTIAL EQUATIONS

EXERCISE 1.8

Problems solved by;

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Differential Eqs are used for finding curves that intersect given curves at Right Angles, A task that arises rather often in practice. The new curves are then called the orthogonal Trajectories of the given curves

Remember that angle of intersection of two curves is defined to be the angle b/w the tangents of the curves at point of intersection.

How To Find ORTHOGONAL TRAJECTORIES

Step # 01 : We find a differential equation

$$y' = f(x, y). \quad \text{--- (1)}$$

for which the given curves are solution curves.

Step # 02 : We write down the diff equations of the orthogonal trajectories to be found, which is.

$$y' = -\frac{1}{f(x, y)}. \quad \text{--- (2)}$$

Two lines are \perp iff $m_1 m_2 = -1 \Rightarrow m_1 = -\frac{1}{m_2}$.

Step # 03 : Solve the differential equation (2)

Note that the differential eq (1) has infinitely many solutions curves, one for each value of the arbitrary constant c in its general solution. Hence we can write this family of curves as

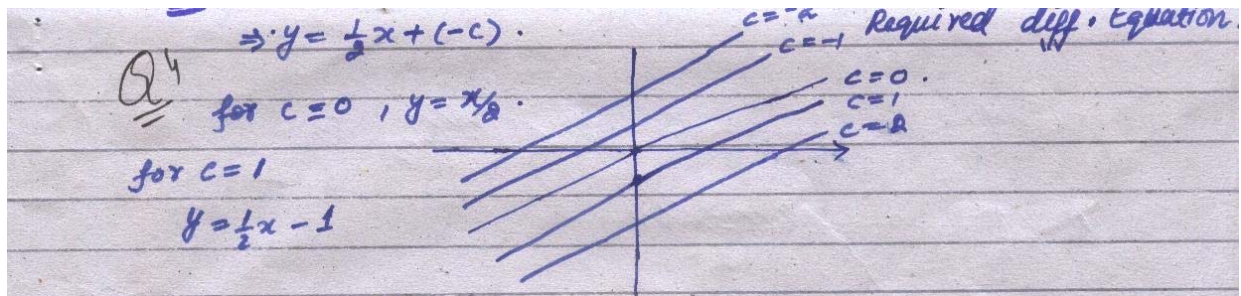
$$F(x, y, c) = 0 \quad \text{--- (3)}$$

e.g, a family of circles

$$x^2 + y^2 = c \text{ can be written as}$$

$$F(x, y, c) = x^2 + y^2 - c = 0$$

This is called a one-parameter family of curves and c is called the parameter of the family.



Q5 $y = ce^{2x} \Rightarrow c = y/e^{2x}$.

$$y' = 2ce^{2x} = 2\left(\frac{y}{e^{2x}}\right)e^{2x} = 2y.$$

$\Rightarrow y' = 2y$. Required differential equation.

Q6 $y = \tan(x+c) \Rightarrow c = \tan^{-1}y - x = C$

$$\Rightarrow y' = \sec^2(x+C) -$$

$$\Rightarrow y' = \sec^2\left(x + (\tan^{-1}(y) - x)\right).$$

Required differential equation.

Q8. $y = cx^4$

$$\Rightarrow y' = 4cx^3 = 4\left(\frac{y}{x^4}\right)(x^3) = \frac{4y}{x} \Rightarrow c = y/x^4.$$

Q9. $c^2x^2 + y^2 = c^2$

$$\Rightarrow c^2x^2 - c^2 = -y^2 \Rightarrow c^2(x^2 - 1) = -y^2.$$

$$\Rightarrow c^2 = -\frac{y^2}{x^2 - 1} \Rightarrow c^2 = \frac{y^2}{1 - x^2}.$$

$c^2x^2 + y^2 = c^2$ diff w.r.t x .

$$\Rightarrow 2c^2x + 2yy' = 0 \Rightarrow y' = -\frac{2c^2x}{2y} = -\frac{c^2x}{y}.$$

$$\Rightarrow y' = -\left(\frac{y^2}{1-x^2}\right)\left(\frac{x}{y}\right) = -\frac{xy}{(1-x^2)} = \frac{xy}{x^2-1}$$

Ans

ORTHOGONAL TRAJECTORIES

Find the orthogonal Trajectories. Plot and sketch some curves and Trajectories.

Q10 $y = ce^{x^2}$.

$$\Rightarrow y' = 2cx e^{x^2}. \text{ slope of the given curve.}$$

$$y' = -\frac{1}{2cx^2} \text{, putting } c = 0/e^x \Rightarrow 0 \text{ and } 2yx$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2cy} \Rightarrow \text{Solve } \int \frac{1}{2cy} dy = \int \frac{1}{2cx^2} dx$$

$$\frac{1}{2} \ln y = -\frac{1}{2c} \ln x + \text{const.}$$

$$\ln y = -\ln x + \text{const.} \Rightarrow y = \frac{1}{x} \cdot \text{const.}$$

$$\Rightarrow y = \frac{1}{x} \cdot \text{const.}$$

$$\Rightarrow \int 2y dy = \int -\frac{1}{x} dx$$

$$\Rightarrow y^2 = -\ln x + \ln c \Rightarrow y^2 = \ln \frac{c}{x}$$

$$\Rightarrow e^{y^2} = \frac{c}{x} \Rightarrow xe^{y^2} = c \quad \text{Ans}$$

$$(11) y = ce^{-x} \Rightarrow c = y/e^{-x}$$

$$\Rightarrow y' = -ce^{-x} \Rightarrow y' = -(y/e^{-x})(e^{-x}) \Rightarrow y' = -y$$

hence the slope of the line tangent to the curve which is perpendicular to the line tangent to the given curve is.

$$y' = -\left(-\frac{1}{y}\right) \Rightarrow y' = \frac{1}{y}$$

$$\Rightarrow \int y dy = \int \frac{1}{y} dy \Rightarrow \frac{y^2}{2} = \ln x + \ln c$$

$$\Rightarrow y^2 = 2 \ln cx \quad \text{Ans} \Rightarrow y = \sqrt{2 \ln cx} \quad \text{Ans}$$

$$(12) y = \ln|x| + c \Rightarrow c = y - \ln x$$

$$\Rightarrow y' = \frac{1}{x} + 0 \Rightarrow y' = \frac{1}{x}$$

hence the slope of the tangent to the curve which is \perp to the given curve is.

$$y' = -\frac{1}{1/x} \Rightarrow y' = -x$$

$$\Rightarrow \int dy = -\int x dx$$

$$\Rightarrow y = -\frac{x^2}{2} + c \quad \text{Ans}$$

Q13

slope of the given curve.

Now the slope of the curve \perp to the given curve is.

$$y' = -\frac{1}{\frac{1}{2x}} \Rightarrow y' = -\frac{2x}{y}$$

and thus the required solution could be obtained by

$$\int y dy = -2 \int x dx \Rightarrow \frac{y^2}{2} = -x^2 + C^*$$

$$\Rightarrow \frac{y^2}{2} = -x^2 + C^* \quad \underline{\text{Ans}}$$

Q14 $y = \sqrt{x+C}$

$$\Rightarrow y' = \frac{1}{2\sqrt{x+C}} \Rightarrow y' = \frac{1}{2\sqrt{x+y^2-x}} \quad \because C = y^2 - x$$

$$\Rightarrow y' = \frac{1}{2y} \quad \text{slope of the line tangent to the given curve.}$$

Now the slope of the curve \perp to given curve is.

$$y' = -\frac{1}{\frac{1}{2y}} = -2y$$

$$\Rightarrow y' = -2y \Rightarrow \int \frac{dy}{y} = \int -2 dx$$

$$\Rightarrow \ln y = -2x + C^* \Rightarrow y = Ce^{-2x} \quad \underline{\text{Ans}}$$

Q15

$y = cx^{3/2}$

$$\Rightarrow y' = \frac{3}{2}cx^{1/2} \Rightarrow y' = \frac{3}{2}\left(\frac{y}{x^{3/2}}\right)(x^{1/2})$$

$$\Rightarrow y' = \frac{3}{2}\left(\frac{y}{x}\right) \quad \text{slope of the given curve.}$$

hence the slope of the curve \perp to the given curve is.

$$y' = -\frac{1}{\frac{3}{2}\left(\frac{y}{x}\right)} = -\frac{2}{3}\left(\frac{x}{y}\right)$$

$$\Rightarrow \int 3y dy = \int -2x dx \Rightarrow \frac{3}{2}y^2 = -x^2 + C^*$$

$$\Rightarrow 3y^2 = -2x^2 + C^* \quad \underline{\text{Ans}}$$

this represents the slope of the given curve. And the slope of the curve \perp to the given curve is.

$$y' = -\left(-\frac{1}{y/x}\right) = x/y$$

$$\Rightarrow \int y dy = \int x dx \Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + \frac{C}{2}$$

$$\Rightarrow x^2 + C = y^2 \quad \text{Ans}$$

Q#17 $y = C/x^2$

$$\Rightarrow y' = -\frac{2C}{x^3} = -\frac{2(yx^2)}{x^3} = -2y/x$$

slope of the given curve.

hence the slope of the curve \perp to the given curve is.

$$y' = -\left(-\frac{1}{2y/x}\right) = +\frac{x}{2y}$$

$$\Rightarrow \int 2y dy = \int x dx \Rightarrow y^2 = \frac{x^2}{2} + C \quad \text{Ans}$$

Q#18 $(x-c)^2 + y^2 = c^2$ — (1)

$$\Rightarrow 2(x-c) + 2yy' = 0$$

$$\Rightarrow y' = -\frac{2(x-c)}{2y} = -\frac{(x-c)}{y} \quad \text{--- (2)}$$

from (1) $x^2 - 2cx + c^2 + y^2 = c^2$

$$\Rightarrow \frac{x^2 + y^2}{2x} = c \quad \text{putting in (2)}$$

$$\Rightarrow y' = -\frac{(x - \frac{x^2 + y^2}{2x})}{y} = -\frac{(2x^2 - x^2 - y^2)}{2xy}$$

$$\Rightarrow y' = -\frac{(x^2 - y^2)}{2xy} \Rightarrow y' = \frac{y}{2x} - \frac{x}{2y} \quad \text{--- (3)}$$

slope of the given curve.

hence the slope of the curve \perp to the given curve is.

$$y' = -\left(\frac{1}{\frac{y}{2x} - \frac{x}{2y}}\right) \Rightarrow y' = -\left(\frac{2xy^2 - 2x^2}{y^2 - x^2}\right)$$

$$\Rightarrow y' = \frac{4xy}{2x^2 - 2y^2} \Rightarrow y' = \frac{2xy}{x^2 - y^2}$$

$$\Rightarrow \frac{1}{y'} = \frac{x}{\frac{\partial x}{\partial y}} - \frac{y}{\frac{\partial y}{\partial x}} = \frac{x}{\frac{\partial x}{\partial y}} - \frac{y}{\frac{\partial y}{\partial x}}$$

$$\Rightarrow \frac{dx}{dy} = \frac{1}{2} \left(\frac{x}{y} - \frac{y}{x} \right).$$

$$\text{let } y/x = u \Rightarrow y = ux \Rightarrow y' = u'x + u$$

$$\Rightarrow y = ux \Rightarrow y' = u'x + u$$

$$\Rightarrow \frac{1}{y'} = \frac{1}{xu' + u}$$

$$\frac{1}{xu' + u} = \frac{1}{2} \left(\frac{1}{u} - u \right) \Rightarrow \frac{1}{xu' + u} = \frac{1}{2} \left(\frac{1-u^2}{u} \right)$$

$$\Rightarrow \frac{2u}{1-u^2} = xu' + u$$

$$\Rightarrow \frac{2u}{1-u^2} - u = xu' \Rightarrow \frac{2u - u(1-u^2)}{1-u^2} = xu'$$

$$\Rightarrow \frac{2u - u + u^3}{1-u^2} = xu'$$

$$y' = \frac{\partial xy}{x^2 - y^2}$$

$$\Rightarrow x' = \frac{x^2 - y^2}{\partial xy} = \frac{x}{\partial y} - \frac{y}{\partial x}$$

$$\text{Now let } u = x/y \Rightarrow x = uy \Rightarrow x' = u'y + u$$

$$= u'y + u = \frac{u}{2} - \frac{1}{2u}$$

$$\Rightarrow u'y = -\frac{u}{2} - \frac{1}{2u} \Rightarrow u'y = -\frac{1}{2} \left(u + \frac{1}{u} \right) = -\frac{1}{2} \left(\frac{u^2 + 1}{u} \right)$$

$$\Rightarrow \frac{u u'}{(u^2 + 1)} = -\frac{1}{2} \Rightarrow \int \frac{u du}{(u^2 + 1)} = -\frac{1}{2} \int \frac{1}{u} du$$

$$\Rightarrow \frac{1}{2} \ln(u^2 + 1) = -\frac{1}{2} \ln y + \ln c$$

$$\Rightarrow \frac{(u^2 + 1)^{1/2}}{c} = y^{-1/2} \Rightarrow (u^2 + 1) = c^2 y$$

$$\Rightarrow \left(\frac{x^2}{y^2} + 1 \right) = c^2 y \Rightarrow x^2 - y^2 = \frac{c^2 y^2}{y}$$

$$\Rightarrow x^2 - y^2 = c^2 y$$

$$\Rightarrow x^2 - c^2 y - y^2 = 0$$